Cosmological perturbations in the Palatini formulation of modified gravity

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Cosmology in extended theories of gravity is considered assuming the Palatini variational principle, for which the metric and connection are independent variables. The field equations are derived to linear order in perturbations about the homogeneous and isotropic but possibly spatially curved background. The results are presented in a unified form applicable to a broad class of gravity theories allowing arbitrary scalar-tensor couplings and nonlinear dependence on the Ricci scalar in the gravitational action. The gauge-ready formalism exploited here makes it possible to obtain the equations immediately in any of the commonly used gauges. Of the three type of perturbations, the main attention is on the scalar modes responsible for the cosmic large-scale structure. Evolution equations are derived for perturbations in a late universe filled with cold dark matter and accelerated by curvature corrections. Such corrections are found to induce effective pressure gradients which are problematical in the formation of large-scale structure. This is demonstrated by analytic solutions in a particular case. A physical equivalence between scalar-tensor theories in metric and in Palatini formalisms is pointed out.

I. INTRODUCTION

The observed acceleration of the universe[1, 2, 3, 4] might indicate that our understanding of gravity breaks down at cosmological distances. In the context of general relativity, what is needed is some sort of fluid with large negative pressure to accelerate the expansion. In its simplest form this fluid is the cosmological constant[5], equivalent to a time-independent vacuum energy density. But there is a problem with the cosmological constant, namely that according to present knowledge of quantum field theory, one would expect this vacuum energy density to be larger than observed by an enormous factor. One approach to this problem is to assume that due to a symmetry the cosmological constant is zero, and in its stead a dynamical fluid, generically called dark energy[6], provides the negative pressure. A plethora of different dark energy models have been proposed, but none without some problems of its own.

As an alternative to energetics of unknown fluids, it has been proposed that the cosmic speed-up could stem from modifications to general relativity[7, 8, 9, 10, 11]. Such modifications can come into play when the gravitational action contains other curvature invariants apart from the standard Einstein-Hilbert term. Then general relativity can be considered as a limit of a hypothetical more general theory. In fact suggestions for such a theory can be found from fundamental physics. Quantization on curved spacetimes has been found long ago to require extension of the Einstein-Hilbert scheme by addition of higher-order curvature terms[12], and it has been also shown that the dominance of these terms at high energies could be the cause of the early inflationary period in the universe[13, 14, 15]. Similarly, corrective terms that become important at small curvature can lead to late-time effects seen in the cosmic expansion as an effective dark energy[16, 17, 18, 19, 20, 21, 22]. It has been shown that such curvature terms can arise from compactification of time-dependent extra dimensions in string/M-theory[23].

Once the gravitational action is nonlinear in R, the question which variational principle to apply becomes relevant. The Palatini variation of a nonlinear gravity action leads to a different theory than the metric variation. The metric variation of extended gravity theories result in fourth order differential equations which are difficult to analyze in practice. The Palatini formulation, in which the connection is treated as an independent variable[24] is more tractable than the metric one, and it can also in general exhibit better stability properties[25, 26], since it yields the modified field equations as a second-order differential system. Mathematical convenience does not of course prove that the Palatini variation would be the fundamentally correct procedure. However, this possibility might be interesting also according to some theoretical prejudices. The second-order nature of the Palatini formulation is conceptually more reconcilable with better-known physics than the metric alternative, where the action in the beginning contains second

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derivatives of the metric, and in the end one has to specify initial values up to third derivatives to predict the evolution of the system. The doubling of the variational degrees of freedom in the Palatini formulation has an analogy with the Hamiltonian mechanics, where the coordinates and momenta of particles are treated as independent variables[27]. On the more speculative side, it is also interesting that the Palatini scheme of gravity can be recovered in unification of general relativity with topological quantum field theory[28].

Nonlinear Palatini gravity may also pass the Solar system tests[29]. While this result has been questioned[30, 31], the most recent discussions support it[32, 33, 34]. Cosmological background solutions have been given for several choices of the gravitational Lagrangian in the Palatini formalism, including (with some constants α and β) $f(R) = R^\alpha$[35], $f(R) = \alpha \log(\beta R)$[36], $f(R) = R + \alpha R^{-\beta}$[26, 37], $f(R) = R + \alpha R^2$[38], $f(R) = R + \alpha/R + \beta R^2$[39] and $f(R) = R + \alpha/R + \beta R^3$[34]. Such solutions have been tested against observational data, confirming that the late accelerating expansion history can be produced viably in these models[26, 40, 41]. This suggests that one should investigate the cosmological implications of these theories beyond the background order. While a wide variety of models is able to produce a scale factor evolution consistent with observations, considerations of the cosmic microwave background anisotropies and large scale inhomogeneities in cosmic structure may distinguish between different physical assumptions underlying similar or even identical expansion histories[42, 43, 44, 45]. Recently it has been shown that early inflation might also be modelled by nonlinear gravity in the Palatini approach[34], and it will be interesting to see the primordial perturbation spectra such models predict.

The aim of this paper is to set up perturbation equations for generalized gravity in the Palatini approach. Cosmological perturbation theory[46, 47, 48] of modified gravities in the metric approach has been rigorously formulated in the gauge-ready form by Hwang and Noh[49, 50, 51]. Recent extensions of these formulations encompass also kinetic theory[52], tachyon condensation[53] and string corrections[54]. The pioneering works found in these references provide an arsenal of efficient methods on which we draw to derive and analyze the Palatini versions of the perturbed field equations. In particular, we will exploit the gauge-ready formalism where the equations are instantaneously obtainable in any of the convenient gauges employed for cosmological perturbations in the literature. Since in the general case it is not clear *a priori* which gauge choice is the most suitable to simplify the analysis, leaving the temporal gauge unfixed in the equations gives them very advantageous adaptability to different situations.

In Section II we will review briefly the Palatini variational principle. We also point out an equivalence between metric and Palatini formulations of scalar-tensor theories, which is verified in detail in Appendix A. In Section III we identify the degrees of freedom in the cosmological metric and the energy-momentum tensor, and derive the corresponding field equations for the three types of perturbations, scalar, vector, and tensor. In Section IV we discuss structure formation in $f(R)$ cosmologies, in particular curvature corrections that become important in the late matter-dominated universe. Section V contains discussion of our results.

II. PALATINI APPROACH TO GENERALIZED GRAVITY

We consider gravity theories represented by the action

$$S = \int d^n x \sqrt{-g} \left[\frac{1}{2} f(R(g_{\mu\nu}, \hat{\Gamma}^\alpha_{\beta\gamma}), \phi) - \frac{1}{2}\omega(\phi)(\partial\phi)^2 - V(\phi) + \mathcal{L}_m(g_{\mu\nu}, \Phi, ...) \right]. \tag{1}$$

Here $\Phi, ...$ are some matter fields. In the Palatini approach one lets the torsionless connection $\hat{\Gamma}^\alpha_{\beta\gamma}$ vary independently of the metric. The Ricci tensor is then defined solely by this connection,

$$R_{\mu\nu} \equiv \hat{\Gamma}^\alpha_{\mu\nu,\alpha} - \hat{\Gamma}^\alpha_{\mu\alpha,\nu} + \hat{\Gamma}^\alpha_{\alpha\lambda}\hat{\Gamma}^\lambda_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\lambda}\hat{\Gamma}^\lambda_{\alpha\nu}, \tag{2}$$

whereas the scalar curvature is given by

$$R \equiv g^{\mu\nu} R_{\mu\nu} \,. \tag{3}$$

The matter energy-momentum tensor is defined as

$$T^{(m)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta(g^{\mu\nu})}. \tag{4}$$

Hereafter we drop the sub- and superscripts m, which stand collectively for all matter except the scalar field ϕ. We consider the second and the third term in the action (1) as the Lagrangian for ϕ. Then the scalar field energy-momentum tensor

$$T^{\mu(\phi)}_\nu = \omega(\nabla^\mu\phi)(\nabla_\nu\phi) - \delta^\mu_\nu \left[\frac{1}{2}\omega(\partial\phi)^2 + V \right]. \tag{5}$$

is found by a similar variation as Eq.(4). The field equations which follow from extremization of the action, Eq.(1), with respect to metric variations, can be written as

$$FR^\mu_\nu - \frac{1}{2} f \delta^\mu_\nu = T^\mu_\nu + T^{\mu(\phi)}_\nu, \tag{6}$$

where we have defined

$$F \equiv \partial f/\partial R. \tag{7}$$

In general relativity, $f = R/8\pi G$ or $(R - 2\Lambda)/8\pi G$, and $F = 1/8\pi G$.

Note that the covariant derivative ∇ in Eq. (5) is taken using the Levi-Civita connection (i.e., the Christoffel symbol) of $g_{\mu\nu}$,

$$\Gamma^\alpha_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda} \right). \tag{8}$$

Thus we have two different connections, the "Palatini" connection $\hat{\Gamma}^\alpha_{\beta\gamma}$, appearing as an independent variable in the action (1), and the Levi-Civita connection $\Gamma^\alpha_{\beta\gamma}$ derived from the metric $g_{\mu\nu}$, and two different covariant derivatives, $\hat{\nabla}$ and ∇, corresponding to these two connections. We denote by $R_{\mu\nu}(g) \equiv \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\alpha\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\alpha\nu}$ the Ricci tensor constructed from the Levi-Civita connection to distinguish it from the Ricci tensor $R_{\mu\nu}$ of Eq. (2). Likewise, we write $R(g) \equiv g^{\mu\nu}R_{\mu\nu}(g)$ to distinguish this scalar curvature, derived solely from the metric, from the scalar curvature of Eq.(3) appearing in the action (1).

By varying the action with respect to $\hat{\Gamma}^\alpha_{\beta\gamma}$, one gets the condition

$$\hat{\nabla}_\alpha \left[\sqrt{-g} g^{\beta\gamma} F \right] = 0, \tag{9}$$

implying that this connection is compatible with the conformal metric

$$\hat{g}_{\mu\nu} \equiv F^{2/(n-2)} g_{\mu\nu}. \tag{10}$$

This connection governs how the tensor $R_{\mu\nu}$ appearing in the action settles itself in order to minimize the action, but it turns out that the connection of Eq.(8) determines the geodesics that freely falling particles follow, since energy momentum is conserved according to this connection[55],

$$\nabla_\mu T^\mu_\nu = 0, \tag{11}$$

whereas in general $\hat{\nabla}_\mu T^\mu_\nu \neq 0$. Therefore one may interpret the Levi-Civita connection as the gravitational field as in Ref.[56]. In any case it is convenient to write the modified field equations in terms of the connection Γ. One finds that the Ricci tensor is

$$R_{\mu\nu} = R_{\mu\nu}(g) + \frac{(n-1)}{(n-2)} \frac{1}{F^2} (\nabla_\mu F)(\nabla_\nu F) - \frac{1}{F} (\nabla_\mu \nabla_\nu F) - \frac{1}{(n-2)} \frac{1}{F} g_{\mu\nu} \Box F. \tag{12}$$

The curvature scalar and Einstein tensor follow straightforwardly,

$$R = R(g) + \frac{n-1}{(n-2)F} \left[-2\Box F + \frac{1}{F} (\partial F)^2 \right], \tag{13}$$

$$G_{\mu\nu} = G_{\mu\nu}(g) + \frac{(n-1)}{(n-2)} \frac{1}{F^2} (\nabla_\mu F)(\nabla_\nu F) - \frac{1}{F} \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) F - \frac{(n-1)}{2(n-2)} \frac{1}{F^2} g_{\mu\nu} (\partial F)^2. \tag{14}$$

From now on we set the spacetime dimension to $n = 4$ and use units $8\pi G \equiv c \equiv 1$. One useful way to write the field equations is in the form of Einstein gravity plus corrections:

$$\begin{aligned} G^\mu_\nu(g) &= T^\mu_\nu + T^{\mu(\phi)}_\nu + (1-F) R^\mu_\nu(g) \\ &- \frac{3}{2F} (\nabla^\mu F)(\nabla_\nu F) + (\nabla^\mu \nabla_\nu F) + \frac{1}{2} \left[(f - R) + (1 - \frac{3}{F}) \Box F + \frac{3}{2F} (\partial F)^2 \right] \delta^\mu_\nu. \end{aligned} \tag{15}$$

Due to Eq.(11) and the Bianchi identity, the correction terms are covariantly conserved according to Γ. In the metric formalism, analogous corrections appear even in vacuum, and it is natural to treat them as an effective additional

fluid[49]. However, here these corrections can be expressed as functions of the trace of the matter energy-momentum tensor $T \equiv g^{\mu\nu}T_{\mu\nu}$ and the scalar field ϕ and its derivatives, and one can view Eq.(15) as general relativity with nonstandard matter couplings. Thus the whole right-hand side may be regarded as an effective matter energy-momentum tensor. In vacuum it reduces to a cosmological constant[57], and in the case of conformal matter (i.e. $T = 0$ [58]) to the usual $T_{\mu\nu}$. One finds R in terms of matter from the trace of the field equation, Eq.(6),

$$FR - 2f = \omega(\partial\phi)^2 - 4V + T. \tag{16}$$

We will refer to this central relation as the structural equation[59, 60].

Now consider some specific cases. Perhaps the simplest nontrivial example is a pure scalar-tensor theory, for which $f = F(\phi)R$. Then the structural equation becomes

$$R = -\frac{\omega\dot{\phi}^2 - 4V + T}{F}. \tag{17}$$

We remark here that this theory is physically equivalent to a similar one in the metric formulation with a modified kinetic term of the scalar field. Namely, with arbitrary $F(\phi)$ and $\omega(\phi)$, the kinetic coefficient $\omega_{\rm metric}$ for the scalar field in the corresponding metric formulation of scalar-tensor theory is

$$\omega_{\rm metric}(\phi) = \omega(\phi) - \frac{3F'^2(\phi)}{2F(\phi)}. \tag{18}$$

We show in Appendix A that the metric and Palatini approaches yield exactly the same field equations and equations of motion when this rescaling of ω is taken into account. Therefore no qualitatively new features, regardless of the scalar field couplings, are introduced in the Palatini approach whenever the gravitational action is linear in R. This correspondence, Eq.(18), enables one to extend the results of studies of scalar-tensor gravity[61] within the metric formalism to the Palatini framework, and may motivate some previously less-investigated forms of these theories.

Consider the case without a non-minimally coupled scalar field, $f = f(R)$. We assume that there exists a solution $R = R(T)$ to the structural equation $FR - 2f = T$. For example, if $f = R - \alpha_0/(3R)$, then

$$R(T) = -\frac{1}{2}\left(T + \sqrt{T^2 + 4\alpha_0}\right). \tag{19}$$

If $f(R) \sim R + \alpha_0 R^2$, we have $R(T) = -T$ as in general relativity. If $f = R^2$, the theory admits only vacuum or conformal matter.

III. COSMOLOGICAL EQUATIONS

A. Definitions

The line-element in the perturbed Friedmann-Robertson-Walker (FRW) spacetime can be written as (see e.g. Ref.[52] for an only slightly different notation)

$$ds^2 = a^2(\eta)\left\{-(1+2\alpha)\,d\eta^2 - 2\left(\beta_{,i} + b_i\right)d\eta dx^i + \left[g^{(3)}_{ij} + 2\left(g^{(3)}_{ij}\varphi + \gamma_{|ij} + c_{(i|j)} + h_{ij}\right)\right]dx^i dx^j\right\}. \tag{20}$$

We characterize the scalar perturbations in a general gauge by the four variables $\alpha, \beta, \varphi, \gamma$. Vector perturbations introduce four more degrees of freedom, the divergenceless 3-vector fields b_i and c_i. Gravitational waves are described by the two free components of the symmetric, transverse and traceless 3-tensor h_{ij}. We have thus decomposed the ten independent components of the symmetric tensor $\delta g_{\mu\nu}$ into three types of perturbations according to their transformation properties under spatial rotations. The comoving spatial background metric $g^{(3)}_{ij}$ reduces to δ_{ij} in a flat universe. The vertical bar indicates a covariant derivative based on the Levi-Civita connection of $g^{(3)}_{ij}$. This metric is used to lower and raise spatial indices $i, j, k \dots$ of the perturbation variables.

The components of the energy-momentum tensor for an imperfect fluid are

$$T^0_0 = -(\bar{\rho} + \delta\rho), \quad T^0_i = -\left(\bar{\rho} + \bar{p}\right)\left(v_{,i} + v^{(v)}_i\right), \quad T^i_j = (\bar{p} + \delta p)\delta^i_j + \Pi^i_j. \tag{21}$$

Here ρ and p are energy density and pressure, and v, $v^{(v)}$ are the scalar and vector velocity perturbations[1], respectively. Background quantities are denoted with an overbar, which we will usually omit when unnecessary. The isotropy of background does not allow anisotropic stress except as a perturbation. This we decompose into the scalar, vector and tensor contributions as

$$\Pi_{ij} \equiv \left(\Pi^{(s)}_{|ij} + \frac{1}{3}\triangle\Pi^{(s)}\right) + \Pi^{(v)}_{(i|j)} + \Pi^{(t)}_{ij}\,, \tag{22}$$

where $\triangle$ stands for the three-space Laplacian based on the Levi-Civita connection of $g^{(3)}_{ij}$. The vector $\Pi^{(v)}_i$ is divergenceless and the tensor $\Pi^{(t)}_{ij}$ is symmetric, transverse, and traceless. The four scalar degrees of freedom for the fluid perturbation are therefore δ, δp, v, and $\Pi^{(s)}$, independent components of the divergenceless vectors $v^{(v)}_i$ and $\Pi^{(v)}_i$ sum up to four and the tensor describing gravitational waves $\Pi^{(t)}_{ij}$ has two independent components.

Some of these degrees of freedom are due to arbitrariness in separating the background from the perturbations. In the gauge-ready formalism[50] one deals with these gauge degrees of freedom by noting that the homogeneity and isotropy of the background space implies invariance of all physical quantities under purely spatial gauge transformations. Therefore one can trade β and γ to the shear perturbation

$$\chi \equiv a(\beta + \dot{\gamma})\,. \tag{23}$$

where an overdot means derivative with respect to the conformal time η. Since both β and γ vary under spatial gauge transformation, they appear only through the spatially invariant linear combination χ in all relevant equations. In addition, one can define the perturbed expansion scalar

$$\kappa \equiv \frac{3}{a}(H\alpha - \dot{\varphi}) - \frac{\triangle}{a^2}\chi\,, \tag{24}$$

Here H is the Hubble parameter defined with respect to conformal time, i.e, the usual Hubble parameter multiplied by the scale factor a. The variable κ is a convenient linear combination, the use of which simplifies some equations, but it is not linearly independent of other perturbations. Only three of the variables α, φ, χ and κ are independent. All of them can be physically interpreted as perturbations of the normal-frame vector[50]. The advantage of using this set of variables is based on the fact that they are spatially gauge-invariant. Thus, writing equations in terms of them, one can conveniently fix the temporal gauge by just setting one of these metric perturbations to zero. The synchronous gauge, corresponding to $\alpha = 0$, is an exception where the gauge mode is removed only up to a constant. For example, the longitudinal (also called the Newtonian or the zero-shear) gauge is the one where $\chi = 0$. One popular gauge is the comoving one, where, instead of setting any of the metric perturbations to zero, one sets the fluid velocity perturbation v in Eq.(21) to $v = 0$. Suitable linear combinations of the above gauge conditions can be considered also.

Similarly, we will exploit the spatially gauge-invariant variable

$$\Psi_i \equiv b_i + \dot{c}_i \tag{25}$$

to characterize vector perturbations of the metric. The tensor perturbation h_{ij} is gauge-invariant by construction.

B. Background

The modified Friedmann equation corresponding to Eq.(6) is

$$3H^2 = \frac{1}{F}\left[a^2\rho + \frac{\omega}{2}\dot{\phi}^2 + a^2 V - 3H\dot{F} - \frac{3}{4F}\dot{F}^2 - \frac{a^2}{2}\left(f - FR\right)\right] - 3K, \tag{26}$$

where K is the curvature of the background space. One should be equipped with a solution to the structural equation Eq.(16), with which to replace the scalar curvature R appearing both explicitly and implicitly (through f, F and

[1] Note that we use the covariant velocity perturbations[62], sometimes denoted as $v \equiv a(V - \beta)$, and $v^{(v)}_i \equiv a(V^{(v)}_i - b_i)$.

their time derivatives) in Eq.(26). For a review of background solutions in several $f(R)$ cases, see[63]. An implicit expression for R follows from Eq.(13):

$$a^2 R = 6\left(\dot{H} + H^2 + K\right) + \frac{3}{F}\left(\ddot{F} + 2H\dot{F} - \frac{1}{2F}\dot{F}^2\right). \tag{27}$$

The matter continuity equation is, as usual,

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{28}$$

Individual matter species, if minimally coupled to gravity and without interactions with other matter species, also satisfy this equation. The Klein-Gordon equation for the scalar field is

$$\ddot{\phi} + 2H\dot{\phi} + \frac{\omega'}{2\omega}\dot{\phi}^2 + \frac{1}{2\omega}\left(2V - f\right)_{,\phi} = 0. \tag{29}$$

C. Field equations for scalar perturbations

First we present the equations that govern the evolution of scalar perturbations defined in the subsection IIIA. From now on we consider variables in the Fourier space. The transformation is simple since at linear order each k-mode evolves independently.

The energy constraint (G^0_0 component of the field equation) in generalized gravity is

$$\begin{aligned}
&\left(2H + \frac{\dot{F}}{F}\right) a\kappa \;+\; \left(6K - 2k^2\right)\varphi + \frac{1}{F}\left(-\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F} + 3H\dot{F}\right)\alpha \\
&\quad = \frac{1}{F}\Bigg\{ - a^2\delta\rho - \omega\dot{\phi}\delta\dot{\phi} - \frac{1}{2}\left[\omega'\dot{\phi}^2 + a^2\left(2V - f\right)_{,\phi}\right]\delta\phi \\
&\qquad + \left[3\left(H^2 + K^2\right) - \frac{3\dot{F}^2}{4F^2} - \frac{a^2}{2}R + k^2\right]\delta F + \left(\frac{3}{2}\frac{\dot{F}}{F} + 3H\right)\delta\dot{F}\Bigg\},
\end{aligned} \tag{30}$$

and the momentum constraint (G^0_i component) is

$$a\kappa - \left(k^2 - 3K\right)\frac{1}{a}\chi + \frac{3\dot{F}}{2F}\alpha = \frac{3}{2F}\left\{a^2\left(\rho + p\right)\frac{v}{k} + \omega\dot{\phi}\delta\phi - \left(H + \frac{3\dot{F}}{2F}\right)\delta F + \delta\dot{F}\right\}. \tag{31}$$

The shear propagation equation ($G^i_j - \frac{1}{3}\delta^i_j G^k_k$ component) reads

$$\frac{1}{a}\dot{\chi} + \left(H + \frac{\dot{F}}{F}\right)\frac{1}{a}\chi - \alpha - \varphi = \frac{1}{F}\left(a^2\Pi^{(s)} + \delta F\right). \tag{32}$$

The Raychaudhuri equation ($G^k_k - G^0_0$ component) is now given by

$$\begin{aligned}
&2a\dot{\kappa} \;+\; \left(4H + \frac{\dot{F}}{F}\right)a\kappa + \left[6\left(\dot{H} - H^2\right) + 6\left(\frac{\ddot{F}}{F} - \frac{\dot{F}^2}{F^2}\right) - 3H\frac{\dot{F}}{F} + \frac{4}{F}\omega\dot{\phi}^2 - 2k^2\right]\alpha + 3\frac{\dot{F}}{F}\dot{\alpha} \\
&\quad = \frac{1}{F}\Bigg\{a^2\left(\delta\rho + 3\delta p\right) + 4\omega\dot{\phi}\delta\dot{\phi} + \left[2\omega'\dot{\phi}^2 + a^2\left(f - 2V\right)_{,\phi}\right]\delta\phi \\
&\qquad + \left(3\frac{\dot{F}^2}{F^2} - a^2R + 6\dot{H} + k^2\right)\delta F - 6\frac{\dot{F}}{F}\delta\dot{F} + 3\delta\ddot{F}\Bigg\}.
\end{aligned} \tag{33}$$

The scalar curvature perturbation can be found by combining previous equations or directly by using Eq.(13),

$$\begin{aligned}
a^2\delta R \;=\; &-2a\dot{\kappa} - \left(8H + 3\frac{\dot{F}}{F}\right)a\kappa + 3\left[2\left(H^2 - \dot{H} - \frac{\ddot{F}}{F}\right) + \frac{\dot{F}^2}{F^2} - \frac{\dot{F}}{F}H + \frac{2}{3}k^2\right]\alpha + 2\left(2k^2 - 6K\right)\varphi - 3\frac{\dot{F}}{F}\dot{\alpha} \\
&+ \frac{3}{F}\left[\left(\frac{\dot{F}^2}{F^2} - 2H\frac{\dot{F}}{F} - \frac{\ddot{F}}{F} + k^2\right)\delta F + \left(2H - \frac{\dot{F}}{F}\right)\delta\dot{F} + \delta\ddot{F}\right].
\end{aligned} \tag{34}$$

Also redundant with Eqs.(30–33) is the perturbed version of the structural equation (G^α_α component)

$$R\delta F + F\delta R - 2\delta f = \frac{2}{a^2}\omega\dot{\phi}\left(\delta\dot{\phi} - \dot{\phi}\alpha\right) + 2(f-2V)_{,\phi}\delta\phi - \delta\rho + 3\delta p. \tag{35}$$

We have $\delta f = F\delta R + f_{,\phi}\delta\phi$, and similarly $\delta F = F_{,R}\delta R + F_{,\phi}\delta\phi$. However, these functions can also be expressed in a form proportional to matter perturbations using the structural equation Eq.(16). Consider the earlier example $f = R - \alpha_0/(3R)$, Eq.(19). There we have

$$\delta f = \left(1 + \frac{\alpha_0}{3R^2(T)}\right)\left(-\frac{1}{2} - \frac{T}{2\sqrt{T^2+4\alpha_0}}\right)\left(-\delta\rho + 3\delta p\right), \tag{36}$$

$$\delta F = \frac{\alpha_0}{3R^3(T)}\left(1 + \frac{1}{\sqrt{T^2+4\alpha_0}}\right)\left(-\delta\rho + 3\delta p\right). \tag{37}$$

It is straightforward to obtain the derivatives $\dot{\delta F}$ and $\ddot{\delta F}$ from Eq.(37). In this manner δR can be related to δT, and one can consider the right-hand sides of the perturbed field equations as matter sources in the modified gravity. However, one possibly appealing gauge condition is to set $\delta F = 0$. This eliminates the complicated effective matter source terms in the right hand side, while leaving some f-dependent terms to modify the evolution of the metric perturbations. In the $f = f(R)$ case this is the gauge where $\delta T = 0$.

The equation of motion for the scalar field is the same as in the metric formulation,

$$\begin{aligned}\delta\ddot{\phi} + \left(\frac{\omega'}{\omega}\dot{\phi} + 2H\right)\delta\dot{\phi} \;&+\; \left[k^2 + \frac{\omega''}{2\omega}\dot{\phi}^2 + \frac{\omega'}{\omega}\left(\ddot{\phi} + 2H\right) + \frac{1}{2\omega}\left(2V - f\right)_{,\phi\phi}\right]\delta\phi \\ &=\; \dot{\phi}\left(\dot{\alpha} + a\kappa\right) + \left(2\ddot{\phi} + H\dot{\phi} + 2\frac{\omega'}{\omega}\dot{\phi}^2\right)\alpha + \frac{1}{2\omega}F_{,\phi}\delta R.\end{aligned} \tag{38}$$

However, now the interpretation of R appearing on both sides of Eq.(38) is different. The minimally coupled individual matter species obey the standard continuity and Euler equations,

$$\delta\dot{\rho} + 3H\left(\delta\rho + \delta p\right) = \left(\rho + p\right)\left(-kv + a\kappa - 3H\alpha\right), \tag{39}$$

$$\frac{1}{a^4}\left[a^4(\rho+p)v\right]^{\cdot} = k\left(\rho+p\right)\alpha + k\delta p - \frac{2}{3}\frac{k^2-3K}{k}\Pi^{(s)}, \tag{40}$$

since matter fields in the action, Eq.(1), decouple from the connection.

D. Vector and tensor perturbations

Vector and tensor modes are simpler in an FRW spacetime than scalar perturbations. This is true especially in $f(R,\phi)$ gravity, since neither R nor ϕ has vector or tensor components, regardless of whether we are in the Palatini or metric formulation. Therefore we can employ the analogous results found in the metric case, see for example[52]. The matter quantities in this subsection do not involve contributions from the scalar field, since it cannot generate vector or tensor perturbations.

The equations governing the evolution of rotational perturbations are

$$\frac{k^2-2K}{2a^2}\Psi_i = \frac{1}{F}(\rho+p)v_i^{(v)}, \tag{41}$$

$$\frac{1}{a^2}\left[a^4(\rho+p)v_i^{(v)}\right]^{\cdot} = -\frac{k^2-2K}{2k^2}\Pi_i^{(v)}. \tag{42}$$

The first is the field equation (the G^0_i component), the second one the conservation equation. The only difference to general relativity here is the appearance of the prefactor $1/F$ modulating the amplitude of Ψ_i. For example, one can see that the angular momentum of a perfect fluid $\sim a^4(\rho+p)v_i^{(v)}$ is conserved in an expanding universe.

The gravitational wave is governed by the equation

$$\ddot{h}^i_j + \left(2H + \frac{\dot{F}}{F}\right)\dot{h}^i_j + \left(k^2 + 2K\right)h^i_j = \frac{a^2}{F}\Pi^{i\,(t)}_j. \tag{43}$$

Modifications to general relativity appear here as an additional damping term and a similar modulating of the source term as for the rotation. In the absence of anisotropic stresses, only the former has an effect. For a flat universe, an integral solution exists for the superhorizon scales,

$$h^i_j(\eta, k) = A^i_j - B^i_j \int^\eta \frac{1}{a^2 F} d\eta, \tag{44}$$

where A^i_j and B^i_j are constants for each k-mode. Thus only the decaying solution is modified when $f \neq R$.

IV. STRUCTURE FORMATION IN $f(R)$ COSMOLOGIES

A. Inhomogeneities in late universe

Now we are in a position to consider formation of structure in cosmologies derived from the Palatini approach to generalized gravity. As discussed in the introduction, it is of particular interest to uncover whether curvature corrections driving the late acceleration will predict testable features in the CMB or large scale structure. Therefore we will consider $f(R)$ theories in a flat universe dominated by pressureless cold dark matter. There is no dark energy, since we consider modified gravity as its alternative.

The evolution of matter perturbations is governed by the equations (39) and (40), which for pressureless matter reduce to

$$\dot{\delta} = -kv + a\kappa - 3H\alpha, \quad \dot{v} = -Hv + k\alpha. \tag{45}$$

We will work first in the uniform density gauge (indicated by the subscript $_\delta$), where $\delta_\delta = 0$. Now also δf_δ and δF_δ together with their derivatives vanish, considerably simplifying the analysis. (This of course does not mean that the matter perturbation disappears, it is just carried by other perturbation quantities, as the physics is completely independent of gauge choice. We later transfer to the comoving gauge, where the interpretation of the results is more straightforward.)

In the uniform density gauge, manipulation of the field equations (30–33) with the help of the conservation equations (45) then yields an evolution equation for the non-vanishing matter velocity perturbation v_δ,

$$\ddot{v}_\delta = \frac{1}{F(2FH + \dot{F})}\left\{\left[-2F^2(H^2 + 2\dot{H}) + 2\dot{F}^2 - \dot{F}FH - 2\ddot{F}F\right]\dot{v}_\delta - \left[6F^2\dot{H}H - 2\dot{F}^2H + \dot{F}F(\dot{H} + \frac{k^2}{3}) + 2\ddot{F}FH\right]v_\delta\right\}. \tag{46}$$

By solving this relatively simple (depending on the form of $f(R)$) differential equation, one can easily find also the metric perturbations, since they are related to v_δ and its derivatives. These solutions can also be related to solutions in any other gauge by using the gauge transformation properties of the relevant variables. In fact, although the evolution equation (46) is particularly tractable in the uniform-density gauge, it is difficult to physically interpret the significance of modifications to general relativity from this equation.

Therefore we transform this result into the more intuitive comoving gauge (subscript $_v$), which is defined by $v_v = 0$. We cannot of course find a nontrivial equation for v_v, but will instead consider the comoving gauge density perturbation δ_v. By looking at Eq.(45) one sees that this gauge also coincides with the synchronous one at the late matter-dominated stage of the universe, regardless of the form of $f(R)$. By using their gauge transformation properties, we find a convenient relation between the perturbation variables in the different gauges,

$$\delta_v = 3Hv_\delta/k. \tag{47}$$

Using this we can recast Eq.(46) into the form

$$\begin{aligned}\ddot{\delta}_v \;&=\; \frac{1}{3FH^2(2FH + \dot{F})}\Big\{ -3H\left[2FH(FH^2 + \ddot{F}) - 2\dot{F}^2H + \dot{F}F(-2\dot{H} + H^2)\right]\dot{\delta}_v \\ &+ \;\left[6F^2H^2(\ddot{H} - 2\dot{H}H) + 6\dot{F}^2H(H^2 - \dot{H}) + \dot{F}F(3\ddot{H}H - 6\dot{H}^2 - H^2k^2) + 6\ddot{F}FH(\dot{H} - H^2)\right]\delta_v\Big\}.\end{aligned} \tag{48}$$

We have checked that the form of this equation does not change when the assumption $K = 0$ is relaxed.

For the case of general relativity with a cosmological constant, $f(R) = R - 2\Lambda$, Eq.(48) reduces to

$$\ddot{\delta}_v = -H\dot{\delta}_v + \left(\frac{\ddot{H}}{H} - 2\dot{H}\right)\delta_v. \tag{49}$$

If the cosmological constant and cold dark matter are thought of as components of a two-component cosmic fluid, the equation of state of this total fluid is $w_T = -(1 + a^{-3}\rho_0/\Lambda)^{-1}$. The density contrast δ_v^T can then be written as $\delta_T = (1 + w_T)\delta_v$. Inserting these in Eq.(49) gives the familiar

$$\ddot{\delta}_T = (6w_T - 1)H\dot{\delta}_T + \frac{3}{2}[1 + 8w_T - 3w_T^2]H^2\delta_T. \tag{50}$$

Notice that this equation is scale-independent, whereas Eq.(48) involves a k^2 -term, which stays small only at very large scales.

Consider first this large scale limit of Eq.(48). There both the "friction" and the "source" term in Eq.(50) are non-trivially modified. Similar effects appear when the cosmological constant is replaced by dynamical dark energy[44, 64]. Here these modifications depend on the specific form of $f(R)$, and one might recover an evolution of δ_v leading to a large-scale matter power spectrum consistent with observations, by choosing a suitable function of R.

However, there could still be difficulties with the small-scale structure in these models. This is because the gradient term in Eq.(48) acts as the dominant source inside the horizon. The gradient appears since the curvature corrections induce effective pressure fluctuations in the inherently cold dark matter. This may have been anticipated from the field equation (15), where it is seen that the modified Palatini gravity is coupled to derivatives of matter energy momentum. Since the cosmological background is homogeneous, modifications to the Friedmann equation stem only from the time variation of the background density of matter. However, the evolution of inhomogeneities in the universe is inevitably affected by the response of the modified Palatini gravity to spatial variations in the distribution of matter. This is why we have found here an effective sound speed for cold dark matter.

It might be useful to compare the situation with dark energy models. A cosmological constant does not fluctuate, therefore the absence of a gradient term in Eq.(50). On the other hand, quintessence fields fluctuate but their gradients are negligible, since those fields are very smooth inside horizon, although this may not be the case if the quintessence field is coupled with dark matter[55]. The models unifying dark energy and dark matter into a single component, are plagued by a nonzero sound speed at small scales[43, 64]. These unified models are somewhat analogous to the case studied here, since we assume that due to the properties of non-standard gravity, the accelerated expansion of the universe, usually explained by dark energy, is in fact driven only by dark matter. Stability of modified gravities in the metric approach [65, 66] has been explored in de Sitter space using covariant and gauge invariant analysis of cosmological perturbations[67, 68], but it remains to be studied whether similar effects in the evolution of matter inhomogeneities as seen here will appear in the metric approach.

Here, at sufficiently large k, the small scale evolution of δ_v is determined by

$$\ddot{\delta}_v = -\frac{F'(T)\rho}{2F(T) + 3F'(T)\rho}k^2\delta_v, \tag{51}$$

where we have expressed the gradient in Eq.(48) in terms of the background solution for F one finds from the structural equation. It is interesting to note that the gradient (or the effective sound speed) does identically vanish only in the case of Einstein gravity with a cosmological constant. For example, when $f(R) = R - \alpha_0/(3R)$ one has

$$\ddot{\delta}_v = -\left[\frac{2x}{3(1+6x) - (3+8x)\sqrt{1+4x}}\right]k^2\delta_v, \tag{52}$$

where the square bracket term is of the order one when $x \equiv \alpha_0\rho^{-2}$ is, which holds when the curvature and matter energies are both relevant. Indeed the effective pressure fluctuation seems to become troublesome inside the horizon.

B. A particular example: $f(R) \sim R^n$

We demonstrate the general considerations of the previous subsection with a specific choice for the nonlinear Lagrangian, $f(R) \sim R^n$, where $n \neq 0, 2, 3$[2]. Such a case has been studied previously[35, 69] and shown to predict a

[2] The limit $n = 3/2$ appears singular in our equations. This is because at that limit the deceleration parameter vanishes and the conformal Hubble parameter cannot be expressed in the form $H \sim 1/\eta$. When $n < 3/2$, η is positive, and when $n > 3/2$, η is growing from negative

plausible background evolution at late times[41]. In fact the background is simply described by a constant effective equation of state in this model. The Hubble parameter scales as $H^2 \sim a^{2-3/n}$. It is then easy to write it with its derivatives in terms of conformal time,

$$H^2 = \frac{4n^2}{(3-2n)^2}\frac{1}{\eta^2}, \quad \dot{H} = \frac{1}{2}(2-\frac{3}{n})H^2, \quad \ddot{H} = \frac{1}{2}(2-\frac{3}{n})^2 H^3. \tag{53}$$

This solution applies also in the metric formulation[70]. In addition, Palatini variation of Ricci squared gravity, $f \sim (R^{\mu\nu}R_{\mu\nu})^{n/2}$, gives the same expansion rate[69].

Here the scalar curvature is $R = 3(3-n)H^2/(2na^2)$. Since Eq.(48) involves only $\dot{F}/F$ and $\ddot{F}/F$, we need only to know that

$$\dot{F} = \frac{3(1-n)}{n}HF, \quad \ddot{F} = \frac{3(1-n)}{2n^2}(3-4n)H^2F. \tag{54}$$

Substituting these in the general Eq.(48) yields

$$(n-\frac{3}{2})\ddot{\delta}_v = \frac{n}{\eta}\dot{\delta}_v + \left[-\frac{3}{\eta^2} + \frac{(n-1)(n-\frac{3}{2})}{3-n}k^2\right]\delta_v. \tag{55}$$

Inserting a power law ansatz $\delta_v \sim a^m$ one finds that in the large-scale limit $k=0$ the solutions are $m = -1/(6n)$ and $m = 3/n - 2$. When n is positive, the former is the decaying solution, and the latter gives the growing mode, which reduces to the standard matter-dominated solution $\delta \sim a$ when $n = 1$. The growth rate reduces when n increases, reflecting the fact that the background expansion is sped up. In the limit $n = 3/2$, when $\ddot{a} = 0$, then also $\dot{\delta}_v = 0$. For larger values of n the large-scale inhomogeneities in fact begin to smooth out, instead of just their growth slowing down.

In the small-scale limit $k^2\eta^2 \gg 1$, and the gradient term drives the perturbations to exponential growth

$$\delta_v \sim \exp\left(\pm\sqrt{\frac{n-1}{3-n}}k\eta\right), \tag{56}$$

when $1 > n > 3$, or to oscillate,

$$\delta_v \sim \exp\left(\pm i\sqrt{\frac{n-1}{n-3}}k\eta\right), \tag{57}$$

when $n > 3$. Since such features are not observed in the matter power spectrum, these models do not appear to be viable alternatives to dark energy. A quantitative study using existing observational data is under progress[71].

V. DISCUSSION

In the metric formulation nonlinear gravity theories result in rather untractable fourth-order differential equations. In the Palatini formulation the structure of the theory becomes interestingly different. Since there is an algebraic relation between the curvature scalar and the trace of the matter energy-momentum tensor, the Palatini variation yields second-order field equations. In Sec. I we discussed observational and theoretical motivations for studying these theories. In this paper we have demonstrated that even the theory of cosmological perturbations is solvable for the Palatini formulation of modified gravities.

In Sec. II we presented the equations for linear cosmological perturbations in a general $f(R,\phi)$-gravity. We arranged them in a gauge-ready form, from which one can easily adopt the most suitable gauge for the particular problem at hand. We found that evolution of vector and tensor perturbations are modestly modified from general relativity, featuring some f-dependent prefactors. Field equations for scalar modes of perturbations are modified in a more intricate manner. In particular, nonlinear Palatini gravity features nonstandard matter couplings, originating from

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values towards zero. Our results are still well defined at the limit $n = 3/2$. On the other hand, $n = 2$ is a physically singular limit of the theory. This is not seen explicitly in our equations, since factors like $1/(n-2)$ belong to an otherwise irrelevant proportionality constant in the solution for f which we have rescaled away. We have also a singularity at n=3 [35].

perturbations in the trace of the fluid energy momentum tensor and its derivatives. Only in a specific gauge, $\delta T = 0$, these couplings ostensibly disappear. These equations can be almost trivially generalized to $f(R,\phi,X)$ gravity, where $X \equiv (\partial\phi)^2$. For simplicity such a case was not included here, but curvature couplings of scalar field derivatives have been introduced as an approach to the cosmological constant problem[19, 60, 72, 73, 74], and it might be interesting to investigate implications of also these models to cosmological perturbations.

In Sec. III we derived also equations governing the growth of inhomogeneities in the late universe undergoing an acceleration of expansion driven by nonlinear $f(R)$-gravity. We found that an effective pressure gradient term appears in the evolution equations for cold dark matter inhomogeneities. This seems to be problematic for structure formation in such models. Analytical solutions were given for a simple example, $f(R) \sim R^n$, but the problem seems generic to these nonlinear gravities. In the $f(R) = R - \alpha_0/(3R)$ case, Eq.(52) implies that the gradient term will drive the perturbations at sufficiently small scales. It might be difficult to tune the form of $f(R)$ (unless it is trivially close to $f(R) = R - 2\Lambda$) in such a way that the gradient term Eq.(51) would be small enough inside the horizon not to affect the linear matter power spectrum. However, an extensive study of the constraints on the form and parameters of the function f in the action (1) is left for forthcoming studies.

APPENDIX A: THE EQUIVALENT METRIC FORMULATION OF A SCALAR-TENSOR THEORY

In n dimensions Eq.(18) generalizes to

$$\omega_{metric}(\phi) = \omega(\phi) - \frac{(n-1)}{(n-2)}\frac{F'^2(\phi)}{F(\phi)}. \tag{A1}$$

Let us therefore consider the action

$$S = \int d^n x\sqrt{-g}\left\{\frac{1}{2}F(\phi)R(g) - \frac{1}{2}\left[\omega(\phi) - \frac{(n-1)}{(n-2)}\frac{F'(\phi)^2}{F(\phi)}\right](\partial\phi)^2 - V(\phi) + \mathcal{L}_m(g_{\mu\nu},\Phi,...)\right\}. \tag{A2}$$

Variations of the metric lead to the field equations

$$\begin{aligned} FR^\mu_\nu(g) - \frac{1}{2}FR(g) &= T^\mu_\nu + (\nabla^\mu\nabla_\nu - \delta^\mu_\nu\Box)F \\ &+ \left[\omega - \frac{(n-1)}{(n-2)}\frac{F'^2}{F}\right](\nabla^\mu\phi)(\nabla_\nu\phi) - \delta^\mu_\nu\left[\left(\omega - \frac{(n-1)}{(n-2)}\frac{F'^2}{F}\right)(\partial\phi)^2 + V\right], \end{aligned} \tag{A3}$$

which can be rewritten as

$$\begin{aligned} G^\mu_\nu(g) &= \frac{1}{F}\left\{T^\mu_\nu + T^{\mu(\phi)}_\nu\right. \\ &+ \left.\left(F'' - \frac{(n-1)}{(n-2)}\frac{F'^2}{F}\right)(\nabla^\mu\phi)(\nabla_\nu\phi) + F'(\nabla^\mu\nabla_\nu\phi) + \delta^\mu_\nu\left[\left(-F'' + \frac{(n-1)}{2(n-2)}\frac{F'^2}{F}\right)(\partial\phi)^2 - F'\Box\phi\right]\right\}, \end{aligned} \tag{A4}$$

where $T^{\mu(\phi)}_\nu$ is given by Eq.(5).

Now let us vary the action of Eq.(1) when $f = F(\phi)R$. We then obtain the Ricci tensor from Eq.(12)

$$R^\mu_\nu = R(g)^\mu_\nu + \frac{1}{F}\left\{\left[\frac{(n-1)}{(n-2)}\frac{F'^2}{F} - F''\right](\nabla^\mu\phi)(\nabla_\nu\phi) + F'(\nabla^\mu\nabla_\nu\phi) - \delta^\mu_\nu\frac{1}{n-2}\left[F''(\partial\phi)^2 + F'\Box\phi\right]\right\}, \tag{A5}$$

and the Ricci scalar from Eq.(13)

$$R = R(g) + \frac{(n-1)}{(n-2)}\frac{1}{F}\left[\left(\frac{F'^2}{F} - 2F''\right)(\partial\phi)^2 - 2F'\Box\phi\right]. \tag{A6}$$

It is then straightforward to plug these into the field equation (6). The result is exactly Eq.(A4), which verifies that the scalar-tensor theory $F(\phi)R$ with the Palatini variation is equivalent to the same theory with the metric variation and the kinetic term rescaled according to Eq.(A1). The equations of motion for the scalar field are built into the field equations[55], they also coincide in these equivalent theories.

APPENDIX B: SCALAR PERTURBATIONS OF THE METRIC

In this Appendix we present some formulae which have been used in the derivation of the field equations. Here all covariant derivatives and curvature variables correspond to the Levi-Civita connection of the metric g of Eq.(20). These results do not depend on the variational principle which one uses. For a more extensive collection of useful formulae for metric perturbations, see appendices in Ref.[75].

The Levi-Civita connection in the perturbed FRW spacetime, Eq.(20), is

$$\begin{aligned}
\Gamma^0_{00} &= H+\dot\alpha\,, \quad \Gamma^0_{0i}=(\alpha-H\beta)_{,i}\,, \quad \Gamma^0_{ij}=g^{(3)}_{ij}\left[H(1-2\alpha+2\varphi)+\dot\varphi\right]+(2H\gamma+\dot\gamma+\beta)_{|ij}\,,\\
\Gamma^i_{00} &= (\alpha-H\beta-\dot\beta)^{,i}\,, \quad \Gamma^i_{0j}=(H+\dot\varphi)\delta^i_j+\dot\gamma^{|i}{}_j\,,\\
\Gamma^i_{jk} &= \Gamma^{i(3)}_{jk}+g^{(3)}_{jk}(H\beta-\varphi)^{,i}+\delta^i_j\varphi_{,k}+\delta^i_k\varphi_{,j}+\gamma_{|j}{}^i{}_k+\gamma_{|k}{}^i{}_j-\gamma_{|j}{}_k{}^i\,.
\end{aligned} \tag{B1}$$

Useful contractions of these are

$$\Gamma^\mu_{\mu 0}=4H+\dot\alpha+3\dot\varphi+\dot\gamma^{|k}{}_k\,, \quad \Gamma^\mu_{\mu i}=\Gamma^{k(3)}_{ki}+\left[\alpha+3\varphi+\gamma^{|k}{}_k\right]_{|i}\,. \tag{B2}$$

Covariant derivatives of a field $\xi(\bar x,t)=\bar\xi(t)+\delta\xi(\bar x,t)$ are then

$$\begin{aligned}
a^2\nabla^0\nabla_0\xi &= -\ddot{\bar\xi}+H\dot{\bar\xi}-\delta\ddot\xi+H\delta\dot\xi+\dot{\bar\xi}\dot\alpha+2\alpha(\ddot{\bar\xi}-H\dot{\bar\xi})\,,\\
a^2\nabla^0\nabla_i\xi &= \left[-\delta\dot\xi+H\delta\xi+\dot{\bar\xi}\alpha\right]_{,i}\,,\\
a^2\nabla^i\nabla_0\xi &= \left[\delta\dot\xi-H\delta\xi-\ddot{\bar\xi}\beta+\dot{\bar\xi}(\alpha+2H\beta)\right]^{,i}\,,\\
a^2\nabla^i\nabla_j\xi &= -\delta^i_jH\dot{\bar\xi}+\left[\delta^i_j(2H\alpha-\dot\varphi)-\frac{1}{a}\chi^{|i}{}_j\right]\dot{\bar\xi}+\delta\xi^{|i}{}_j-\delta^i_jH\delta\dot\xi\,,\\
a^2\Box\xi &= -\ddot{\bar\xi}-2H\dot{\bar\xi}-\delta\ddot\xi-2H\delta\dot\xi+\delta\xi^{|k}\quad k+(2\ddot{\bar\xi}+H\dot{\bar\xi})\alpha+\dot{\bar\xi}(\dot\alpha+a\kappa)\,.
\end{aligned} \tag{B3}$$

The Ricci tensor can be calculated using the "hatless" version of Eq.(2),

$$R(g)_{\mu\nu}\equiv\Gamma^\alpha_{\mu\nu,\alpha}-\Gamma^\alpha_{\mu\alpha,\nu}+\Gamma^\alpha_{\alpha\lambda}\Gamma^\lambda_{\mu\nu}-\Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\alpha\nu}\,, \tag{B4}$$

with Eqs. (B1),

$$\begin{aligned}
a^2R(g)^0_0 &= 3\dot H-a\dot\kappa-2Ha\kappa+3(H^2-\dot H)\alpha-\triangle\alpha\,,\\
a^2R(g)^i_0 &= 2\left[H\alpha-\dot\varphi+(\dot H-H^2)\beta\right]^{,i}\,,\\
a^2R(g)^0_i &= 2\left[-H\alpha+\dot\alpha\right]_{,i}-R(g)^{(3)}_{ik}\beta^{,i},\,,\\
a^2R(g)^i_j &= \delta^i_j(\dot H+2H^2)+R(g)^{i(3)}_j+\delta^i_j\left[\ddot\varphi-H(\dot\alpha-5\dot\varphi)-2(\dot H+2H^2)\alpha-2R^{i(3)}_j\varphi+\Delta(-\varphi+\frac{H}{a}\chi)\right]\\
&+\left[-\alpha-\varphi+\frac{1}{a}\dot\chi+\frac{H}{a}\chi\right]^{|i}{}_j\,.
\end{aligned} \tag{B5}$$

The curvature scalar is

$$a^2R(g)=6(\dot H+H^2)+R(g)^{(3)}-2\left[a\dot\kappa+4Ha\kappa+3(\dot H-H^2)\alpha+(R(g)^{(3)}+2\triangle)\varphi+\triangle\alpha\right]\,. \tag{B6}$$

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